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Non-linear Kelvin-Helmholtz instability

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A non-linear analysis is presented for the stability of a liquid film adjacent to a compressible gas and under the influence of a body force directed either outward from or toward the liquid. The effects of the liquid's surface tension are taken into account. The non-linear Rayleigh–Taylor instability is included as a special case. The analysis considers the case of an inviscid liquid adjacent to a subsonic flow and the case of a very viscous liquid adjacent to a subsonic or a supersonic flow. For a subsonic external flow, it is found that the cut-off wave-number is amplitude dependent in the inviscid case whereas it is amplitude independent in the viscous case. It is found that the non-linear motion of the gas may be stabilizing or destabilizing, whereas the non-linear motion of the liquid is found to be stabilizing in the viscous case. For a supersonic external flow and a viscous liquid, the cut-off wave-number is amplitude dependent. Moreover, unstable disturbances with wave-numbers near the cut-off wave-number do not grow indefinitely with time but achieve a steady-state amplitude.

1. Introduction

The instability of the interface of two superimposed fluids with different densities has been studied extensively. In the absence of convective instabilities, there are three principal instability mechanisms, namely, (*a*) gravity (Rayleigh 1883; Taylor 1950), (*b*) pressure perturbation or Kelvin–Helmholtz instability (Chandrasekhar 1961; Chang & Russell 1965), and (*c*) shear perturbation (Craik 1966; Benjamin 1959; Miles 1962). In this paper we are concerned only with the first two.

In the absence of an external gas, the instability is provided by body forces acting outward from the liquid layer. In this case, the effect of surface tension is to stabilize the motion for wave-numbers greater than a cut-off wave-number k_c, while the effects of viscosity and finite depth are to reduce the amplification rate of unstable modes and cannot, by themselves, stabilize these modes. The effect of non-linearity was shown by Nayfeh (1969) to be destabilizing because both the amplification rate and the cut-off wave-number increase as the disturbance amplitude increases.

The addition of an external gas allows perturbations in the stresses exerted by the gas on the interface due to the appearance of waves. The essence of the Kelvin–Helmholtz mechanism is that the pressure perturbation exerted by the gas does work on the interface. The amount of work done depends on the magnitude and phasing of the pressure with respect to the wave. This phasing depends on whether the external gas is viscous or inviscid, laminar or turbulent, and whether it is incompressible, subsonic or supersonic. If the external gas is inviscid and subsonic (including incompressible), flowing parallel to the undisturbed surface with a uniform mean velocity, the pressure perturbation is 180° out of phase with the surface amplitude. In this case, the gas pushes down at the troughs of the wave and sucks at the crests of the wave, thereby feeding energy to the disturbance in the liquid layer. On the other hand, if the external gas flow is non-uniform, the pressure disturbances lag the wave by less than 180°. Consequently, the axial component of the pressure perturbation can do work on the wave (Miles 1957). A different situation arises if the external flow is supersonic. In this case, the pressure perturbation is in phase with the wave slope, giving rise to maximum energy transfer from the gas to the liquid through supersonic drag. Chang & Russell (1965), and Willson & Chang (1967) found instability at all wave-numbers in the inviscid limit for supersonic flow with a uniform mean velocity parallel to the undisturbed surface, even in the presence of surface tension. However, they showed that the introduction of viscosity gives a cut-off wave-number above which disturbances are stable.

The purpose of the present paper is to study the combined effect of pressure perturbations and body forces on the non-linear stability of finite-depth liquid layers, taking surface tension into account. One surface of the liquid is taken to be adjacent to a solid body, while the second surface is taken to be adjacent to a subsonic or supersonic inviscid gas stream flowing parallel to the solid surface. Hence, the effects of shear perturbations and non-uniform velocity profiles will not be considered. The liquid motion is assumed also to be two-dimensional and initially quiescent. Subsequent to the submittal of this paper, Drazin (1970) analyzed the non-linear Kelvin-Helmholtz instability of two parallel horizontal streams of inviscid incompressible fluids.

2. Case of inviscid liquid and inviscid subsonic gas

We assume that the motions of both the liquid and the gas are represented by potential functions, and we consider a standing or travelling sinusoidal disturbance with amplitude a and wave-number k'. Distances and time are made dimensionless using $1/k'$ and $(gk')^{-\frac{1}{2}}$, respectively, where g is the body force normal to the undisturbed gas/liquid interface and directed from the liquid to the gas. The gas density, ρ_g, is assumed to be very small compared to that of the liquid, ρ, so that the gas body force can be neglected. The transient motion of the gas is neglected because the gas velocity, U_g, is much larger than the wave velocity for the cases we consider.

A Cartesian co-ordinate system is introduced such that the x and y axes are in and normal to the undisturbed liquid/gas interface. The potential functions representing the motion of the liquid and the gas are taken to be

$$g^{\frac{1}{2}}k'^{-\frac{3}{2}}\varphi(x,y,t), \quad U_g[x+\Phi(x,y,t)]/k',$$

where the dimensionless functions, φ and Φ satisfy

$$\nabla^2\varphi = 0, \quad -h \leqslant y < \eta \tag{2.1}$$

and (see Van Dyke 1964)

$$\Phi_{yy} + m^2\Phi_{xx} = M^2[\tfrac{1}{2}(\gamma-1)(2\Phi_x + \Phi_x^2 + \Phi_y^2)(\Phi_{xx} + \Phi_{yy}) + (2\Phi_x + \Phi_x^2)\Phi_{xx} + 2(1+\Phi_x)\Phi_y\Phi_{xy} + \Phi_y^2\Phi_{yy}] \quad (\eta \leqslant y < \infty), \tag{2.2}$$

for $-\infty < x < \infty$, where h is the depth of the liquid layer, $\eta(x,t)$ is the elevation of the wave above the undisturbed interface, M is the gas Mach number, and

$$m^2 = 1 - M^2.$$

At the solid/liquid interface, the normal velocity vanishes, i.e.

$$\varphi_y(x, -h, t) = 0. \tag{2.3}$$

Since Φ_y vanishes away from the liquid surface,

$$\Phi_y(x, \infty, t) = 0. \tag{2.4}$$

At the liquid/gas interface, the normal components of the gas and the liquid velocities are equal to each other and to the normal velocity of the interface itself, i.e.

$$\eta_t + \varphi_y = \varphi_x\eta_x \quad \text{at} \quad y = \eta, \tag{2.5}$$

$$\eta_x - \Phi_y = -\eta_x\Phi_x \quad \text{at} \quad y = \eta. \tag{2.6}$$

Moreover, the balance of normal forces on this interface gives

$$-\eta - \varphi_t + \tfrac{1}{2}(\varphi_x^2 + \varphi_y^2) = k^2\eta_{xx}(1+\eta_x^2)^{-\frac{3}{2}} - \tfrac{1}{2}mk\chi C_p \quad \text{at} \quad y = h, \tag{2.7}$$

where C_p is the pressure perturbation coefficient exerted by the gas on the interface due to the appearance of waves, and it is given by (Liepmann & Roshko 1957)

$$C_p = (2/\gamma M^2)\{[1 - \tfrac{1}{2}(\gamma-1)M^2(2\Phi_x + \Phi_x^2 + \Phi_y^2)]^{\gamma/(\gamma-1)} - 1\}, \tag{2.8}$$

with γ the specific heat ratio of the gas. The parameters k and χ are defined by

$$k = k'/k'_c, \quad k'_c = (\rho g/T)^{\frac{1}{2}}, \quad \chi = \rho_g U_g^2 k'_c/m\rho g,$$

where T is the surface tension and k'_c is the linear cut-off wave-number in the absence of the gas motion.

The initial conditions are taken to be

$$\eta(x, 0) = \epsilon\cos x, \quad \epsilon = ak', \tag{2.9}$$

$$\eta_t(x, 0) = 0. \tag{2.10}$$

The problem posed by the system of equations (2.1)–(2.10) is essentially a non-linear oscillation problem. Therefore, in order to determine an approximate solution to this system for small but finite ϵ, we use the method of multiple scales (Nayfeh 1965, 1968) by introducing the time scales

$$T_0 = t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t. \tag{2.11}$$

Consequently, the time derivative is transformed according to

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \dots. \tag{2.12}$$

Moreover, we assume expansions of the form

$$\eta(x,t;\epsilon) = \sum_{n=1}^{3} \epsilon^n \eta_n(x, T_0, T_1, T_2) + O(\epsilon^4), \tag{2.13}$$

$$\varphi(x,y,t;\epsilon) = \sum_{n=1}^{3} \epsilon^n \varphi_n(x, y, T_0, T_1, T_2) + O(\epsilon^4), \tag{2.14}$$

$$\Phi(x,y,t;\epsilon) = \sum_{n=1}^{3} \epsilon^n \Phi_n(x, y, T_0, T_1, T_2) + O(\epsilon^4). \tag{2.15}$$

Since (2.1), (2.3), and (2.4) are linear, each φ_n satisfies (2.1) and (2.3), while each Φ_n satisfies (2.4).

Substituting the above expansions into the remaining equations and equating coefficients of like powers of ϵ, we get

Order ϵ:

$$m^2 \frac{\partial^2 \Phi_1}{\partial x^2} + \frac{\partial^2 \Phi_1}{\partial y^2} = 0, \tag{2.16}$$

$$\frac{\partial \eta_1}{\partial T_0} + \frac{\partial \varphi_1}{\partial y} = 0 \quad \text{at} \quad y = 0, \tag{2.17}$$

$$\frac{\partial \eta_1}{\partial x} - \frac{\partial \Phi_1}{\partial y} = 0 \quad \text{at} \quad y = 0, \tag{2.18}$$

$$-\eta_1 - \frac{\partial \varphi_1}{\partial T_0} - k^2 \frac{\partial^2 \eta_1}{\partial x^2} - mk\chi \frac{\partial \Phi_1}{\partial x} = 0 \quad \text{at} \quad y = 0, \tag{2.19}$$

$$\eta_1(x,0,0,0) = \cos x, \quad \frac{\partial \eta_1}{\partial T_0}(x,0,0,0) = 0. \tag{2.20}$$

Order ϵ^2:

$$m^2 \frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial^2 \Phi_2}{\partial y^2} = M^2 \left[(\gamma+1) \frac{\partial \Phi_1}{\partial x} \frac{\partial^2 \Phi_1}{\partial x^2} + (\gamma - 1) \frac{\partial \Phi_1}{\partial x} \frac{\partial^2 \Phi_1}{\partial y^2} + 2 \frac{\partial \Phi_1}{\partial y} \frac{\partial^2 \Phi_1}{\partial x \partial y} \right], \tag{2.21}$$

$$\frac{\partial \eta_2}{\partial T_0} + \frac{\partial \varphi_2}{\partial y} = \frac{\partial \varphi_1}{\partial x} \frac{\partial \eta_1}{\partial x} - \eta_1 \frac{\partial^2 \varphi_1}{\partial y^2} - \frac{\partial \eta_1}{\partial T_1} \quad \text{at} \quad y = 0, \tag{2.22}$$

$$\frac{\partial \eta_2}{\partial x} - \frac{\partial \Phi_2}{\partial y} = -\frac{\partial \eta_1}{\partial x} \frac{\partial \Phi_1}{\partial x} + \eta_1 \frac{\partial^2 \Phi_1}{\partial y^2} \quad \text{at} \quad y = 0, \tag{2.23}$$

$$\begin{aligned} -\eta_2 - \frac{\partial \varphi_2}{\partial T_0} - k^2 \frac{\partial^2 \eta_2}{\partial x^2} - mk\chi \frac{\partial \Phi_2}{\partial x} = & -\frac{1}{2} \left(\frac{\partial \varphi_1}{\partial x} \right)^2 - \frac{1}{2} \left(\frac{\partial \varphi_1}{\partial y} \right)^2 + \eta_1 \frac{\partial^2 \varphi_1}{\partial y \partial T_0} \\ & + mk\chi \eta_1 \frac{\partial^2 \Phi_1}{\partial x \partial y} + \tfrac{1}{2} m^3 k \chi \left(\frac{\partial \Phi_1}{\partial x} \right)^2 \\ & + \tfrac{1}{2} mk\chi \left(\frac{\partial \Phi_1}{\partial y} \right)^2 + \frac{\partial \varphi_1}{\partial T_1} \quad \text{at} \quad y = 0, \end{aligned} \tag{2.24}$$

$$\eta_2 = 0, \quad \frac{\partial \eta_2}{\partial T_0} = -\frac{\partial \eta_1}{\partial T_1} \quad \text{at} \quad T_0 = T_1 = T_2 = 0. \tag{2.25}$$

Order ϵ^3:

$$m^2 \frac{\partial^2 \Phi_3}{\partial x^2} + \frac{\partial^2 \Phi_3}{\partial y^2} = M^2 \left\{ (\gamma+1) \left(\frac{\partial \Phi_1}{\partial x} \frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial \Phi_2}{\partial x} \frac{\partial^2 \Phi_1}{\partial x^2} \right) + (\gamma - 1) \right.$$
$$\times \left(\frac{\partial \Phi_1}{\partial x} \frac{\partial^2 \Phi_2}{\partial y^2} + \frac{\partial \Phi_2}{\partial x} \frac{\partial^2 \Phi_1}{\partial y^2} \right) + 2 \left(\frac{\partial \Phi_1}{\partial y} \frac{\partial^2 \Phi_2}{\partial x \partial y} + \frac{\partial \Phi_2}{\partial y} \frac{\partial^2 \Phi_1}{\partial x \partial y} \right)$$
$$+ \frac{\gamma - 1}{2} \left[\left(\frac{\partial \Phi_1}{\partial x} \right)^2 + \left(\frac{\partial \Phi_1}{\partial y} \right)^2 \right] \left[\frac{\partial^2 \Phi_1}{\partial x^2} + \frac{\partial^2 \Phi_1}{\partial y^2} \right] + \left(\frac{\partial \Phi_1}{\partial x} \right)^2 \frac{\partial^2 \Phi_1}{\partial x^2}$$
$$\left. + 2 \frac{\partial \Phi_1}{\partial x} \frac{\partial \Phi_1}{\partial y} \frac{\partial^2 \Phi_1}{\partial x \partial y} + \left(\frac{\partial \Phi_1}{\partial y} \right)^2 \frac{\partial^2 \Phi_1}{\partial y^2} \right\}, \tag{2.26}$$

$$\frac{\partial \eta_3}{\partial T_0} + \frac{\partial \varphi_3}{\partial y} = \frac{\partial \varphi_1}{\partial x} \frac{\partial \eta_2}{\partial x} + \left(\frac{\partial \varphi_2}{\partial x} + \eta_1 \frac{\partial^2 \varphi_1}{\partial x \partial y} \right) \frac{\partial \eta_1}{\partial x} - \eta_1 \frac{\partial^2 \varphi_2}{\partial y^2} - \tfrac{1}{2} \eta_1^2 \frac{\partial^3 \varphi_1}{\partial y^3}$$
$$- \eta_2 \frac{\partial^2 \varphi_1}{\partial y^2} - \frac{\partial \eta_2}{\partial T_1} - \frac{\partial \eta_1}{\partial T_2} \quad \text{at} \quad y = 0, \tag{2.27}$$

$$\frac{\partial \eta_3}{\partial x} - \frac{\partial \Phi_3}{\partial y} = - \frac{\partial \Phi_1}{\partial x} \frac{\partial \eta_2}{\partial x} - \left(\frac{\partial \Phi_2}{\partial x} + \eta_1 \frac{\partial^2 \Phi_1}{\partial x \partial y} \right) \frac{\partial \eta_1}{\partial x} + \eta_1 \frac{\partial^2 \Phi_2}{\partial y^2} + \tfrac{1}{2} \eta_1^2 \frac{\partial^3 \Phi_1}{\partial y^3}$$
$$+ \eta_2 \frac{\partial^2 \Phi_1}{\partial y^2} \quad \text{at} \quad y = 0, \tag{2.28}$$

$$\eta_3 + \frac{\partial \varphi_3}{\partial T_0} + k^2 \frac{\partial^2 \eta_3}{\partial x^2} + mk\chi \frac{\partial \Phi_3}{\partial x} - \frac{\partial \varphi_1}{\partial x} \frac{\partial \varphi_2}{\partial x} - \frac{\partial \varphi_1}{\partial y} \frac{\partial \varphi_2}{\partial y} - \eta_1 \frac{\partial \varphi_1}{\partial x} \frac{\partial^2 \varphi_1}{\partial x \partial y}$$
$$- \eta_1 \frac{\partial \varphi_1}{\partial y} \frac{\partial^2 \varphi_1}{\partial y^2} - \tfrac{3}{2} k^2 \left(\frac{\partial \eta_1}{\partial x} \right)^2 \frac{\partial^2 \eta_1}{\partial x^2} + \eta_1 \frac{\partial^2 \varphi_2}{\partial y \partial T_0} + \tfrac{1}{2} \eta_1^2 \frac{\partial^3 \varphi_1}{\partial y^2 \partial T_0} + \eta_2 \frac{\partial^2 \varphi_1}{\partial y \partial T_0}$$
$$+ \frac{\partial \varphi_2}{\partial T_1} + \frac{\partial \varphi_1}{\partial T_2} + \eta_1 \frac{\partial^2 \varphi_1}{\partial T_1 \partial y} + mk\chi \, \eta_1 \frac{\partial^2 \Phi_2}{\partial x \partial y} + \tfrac{1}{2} mk\chi \, \eta_1^2 \frac{\partial^3 \Phi_1}{\partial x \partial y^2} + mk\chi \, \eta_2 \frac{\partial^2 \Phi_1}{\partial x \partial y}$$
$$+ m^3 k\chi \left(\frac{\partial \Phi_1}{\partial x} \frac{\partial \Phi_2}{\partial x} + \eta_1 \frac{\partial^2 \Phi_1}{\partial x \partial y} \frac{\partial \Phi_1}{\partial x} \right) + mk\chi \left(\frac{\partial \Phi_1}{\partial y} \frac{\partial \Phi_2}{\partial y} + \eta_1 \frac{\partial^2 \Phi_1}{\partial y^2} \frac{\partial \Phi_1}{\partial y} \right)$$
$$- \tfrac{1}{2} M^2 mk\chi \left[\left(\frac{\partial \Phi_1}{\partial x} \right)^3 + \frac{\partial \Phi_1}{\partial x} \left(\frac{\partial \Phi_1}{\partial y} \right)^2 \right] = 0 \quad \text{at} \quad y = 0, \tag{2.29}$$

$$\eta_3 = 0, \quad \partial \eta_3 / \partial T_0 = - \partial \eta_1 / \partial T_2 - \partial \eta_2 / \partial T_1 \quad \text{at} \quad T_0 = T_1 = T_2 = 0. \tag{2.30}$$

Third-order solutions are given for travelling and standing waves in §§2.1 and 2.2, respectively. As in the non-linear Rayleigh–Taylor instability, these expansions are not valid at or near the cut-off wave-numbers. An expansion valid near these cut-off wave-numbers is presented in §2.3. Moreover, the expansions of §§2.1 and 2.2 have two singularities corresponding to the two-to-one and three-to-one resonances. A second-order expansion is presented in §2.4 for periodic travelling waves for the two-to-one resonance case.

2.1. *Expansion for travelling waves*

The solution of the first-order problem is

$$\eta_1 = \cos\theta, \tag{2.31}$$

$$\varphi_1 = \mu_1 \frac{\cosh(y+h)}{\sinh h} \sin\theta, \tag{2.32}$$

$$\Phi_1 = m^{-1} e^{-my} \sin\theta, \tag{2.33}$$

where
$$\mu_n^2 = n(n^2k^2 - n\chi k - 1)/C_n, \quad C_n^{-1} = \tanh nh, \tag{2.34}$$

$$\theta = x + \mu_1 T_0 + \beta(T_1, T_2), \quad \beta(0,0) = 0. \tag{2.35}$$

The linear solution is obtained if β is taken to be a constant. The solution shows that disturbances are stable or unstable, depending on whether μ_1^2 is positive or negative. The cut-off wave-numbers (wave-numbers corresponding to neutrally stable disturbances) are given by $\mu_1^2 = 0$; that is,

$$k^2 - \chi k - 1 = 0 \tag{2.36}$$

or
$$k_c = \tfrac{1}{2}[\chi \pm (\chi^2 + 4)^{\frac{1}{2}}]. \tag{2.37}$$

Since the wave-numbers are positive, only the positive sign in (2.37) has a physical significance. Equation (2.37) shows that the cut-off wave-number is independent of the liquid depth as in the Rayleigh–Taylor instability. Disturbances are stable if $k > k_c$ and unstable if $k < k_c$.

If the body force is directed from the gas to the liquid, (2.37) is modified to

$$k_c = \tfrac{1}{2}[\chi \pm (\chi^2 - 4)^{\frac{1}{2}}]. \tag{2.38}$$

Hence, there are two cut-off wave-numbers, and disturbances are stable only if their wave-numbers lie outside these two cut-off wave-numbers, otherwise, they are unstable. For $\chi < 2$, the flow is stable for all wavelengths which is the classical Kelvin–Helmholtz problem.

With the above solution, (2.22)–(2.25) become

$$m^2 \frac{\partial^2 \Phi_2}{\partial y^2} + \frac{\partial^2 \Phi_2}{\partial y^2} = -\tfrac{1}{2}(\gamma+1)\frac{M^4}{m^2} e^{-2my} \sin 2\theta, \tag{2.39}$$

$$\frac{\partial \eta_2}{\partial T_0} + \frac{\partial \varphi_2}{\partial y} = -\mu_1 C_1 \sin 2\theta + \frac{\partial \beta_1}{\partial T_1} \sin\theta \quad \text{at} \quad y = 0, \tag{2.40}$$

$$\frac{\partial \eta_2}{\partial x} - \frac{\partial \Phi_2}{\partial y} = \frac{1}{2}\,\frac{1+m^2}{m} \sin 2\theta \quad \text{at} \quad y = 0, \tag{2.41}$$

$$-\eta_2 - \frac{\partial \varphi_2}{\partial T_0} - k^2 \frac{\partial^2 \eta_2}{\partial x^2} - mk\chi \frac{\partial \Phi_2}{\partial x} = \tfrac{1}{4}(1 - C_1^2)\mu_1^2 + \tfrac{1}{4}[(3 - C_1^2)\mu_1^2 - 2mk\chi]\cos 2\theta + \mu_1 C_1 \frac{\partial \beta}{\partial T_1} \cos\theta \quad \text{at} \quad y = 0. \tag{2.42}$$

In order to prevent secular terms from entering into the solution, $\partial\beta/\partial T_1 = 0$, i.e. $\beta = \beta(T_2)$. Hence the solution of the second-order problem is

$$\eta_2 = a_{22}(\cos 2\theta - \cos\theta_2), \tag{2.43}$$

$$\varphi_2 = \tfrac{1}{4}\mu_1^2(C_1^2 - 1)\,T_0 + (\mu_1 b_{22} \sin 2\theta - \tfrac{1}{2}\mu_2 a_{22} \sin\theta_2)\frac{\cosh 2(y+h)}{\sinh 2h}, \tag{2.44}$$

$$\Phi_2 = \left[\tfrac{1}{8}(\gamma+1)\frac{M^4}{m^3} y + \frac{d_{22}}{m}\right] e^{-2my} \sin 2\theta - \frac{a_{22}}{m} e^{-2my} \sin\theta_2, \tag{2.45}$$

where
$$\theta_2 = 2x+\mu_2 T_0, \tag{2.46}$$

$$a_{22} = \left\{\mu_1^2(3-C_1^2-4C_1C_2)+\frac{2k\chi}{m}\left[1+\tfrac{1}{4}(\gamma+1)\frac{M^4}{m^2}\right]\right\}\Big/2C_2(\mu_2^2-4\mu_1^2), \tag{2.47}$$

$$b_{22} = a_{22}-\tfrac{1}{2}C_1, \tag{2.48}$$

$$d_{22} = a_{22}+\frac{1}{4}\frac{1+m^2}{m^2}+\tfrac{1}{16}(\gamma+1)\frac{M^4}{m^3}. \tag{2.49}$$

The first- and second-order solutions determine the right-hand sides of (2.26)–(2.29), and they become

$$m^2\frac{\partial^2\Phi_3}{\partial x^2}+\frac{\partial^2\Phi_3}{\partial y^2} = (p_1+qy)\,e^{-3my}\sin\theta+\text{NSPT}, \tag{2.50}$$

$$\frac{\partial\eta_3}{\partial T_0}+\frac{\partial\varphi_3}{\partial y} = (p_2+\beta')\sin\theta+\text{NSPT}, \tag{2.51}$$

$$\frac{\partial\eta_3}{\partial x}-\frac{\partial\Phi_3}{\partial y} = p_3\sin\theta+\text{NSPT}, \tag{2.52}$$

$$-\eta_3-\frac{\partial\varphi_3}{\partial T_0}-k^2\frac{\partial^2\eta_3}{\partial x^2}-mk\chi\left|\frac{\partial\Phi_3}{\partial x}\right. = (p_4+k\chi p_5+\mu_1C_1\beta')\cos\theta+\text{NSPT}, \tag{2.53}$$

where the p's and q are given in the appendix, NSPT stands for non-secular producing terms, and $\beta' = d\beta/dT_2$. The particular solution of the third-order problem contains secular terms which make η_3/η_1 unbounded as $T_0 \to \infty$. To determine the conditions which must be satisfied for there to be no secular terms, we assume that the particular solution corresponding to these secular producing terms is

$$\eta_3 = 0, \tag{2.54}$$

$$\Phi_3 = \left[\frac{1}{8m^2}\left(p_1+\frac{3}{4}\frac{q}{m}\right)+\frac{1}{8}\frac{q}{m^2}y\right]e^{-3my}\sin\theta+D\,e^{-my}\sin\theta, \tag{2.55}$$

$$\varphi_3 = E\,\frac{\cosh(y+h)}{\sinh h}\sin\theta. \tag{2.56}$$

This particular solution satisfies (2.1) and (2.50). Substituting this solution into (2.51)–(2.53), and equating the coefficients of $\sin\theta$ and $\cos\theta$ on both sides, we get

$$E = p_2+\beta', \tag{2.57}$$

$$mD+\frac{3}{8}\frac{p_1}{m}+\frac{5}{32}\frac{q}{m^2} = p_3, \tag{2.58}$$

$$-\mu_1C_1E-\frac{1}{8}\frac{k\chi}{m}\left(p_1+\frac{3}{4}\frac{q}{m}\right)-mk\chi D = p_4+k\chi p_5+\mu_1C_1\beta'. \tag{2.59}$$

Elimination of E and D from these equations yields

$$\beta' = [(C_1^2+4C_1C_2-3)\,a_{22}+3C_1-2C_1^2C_2]\,\mu_1/4C_1-3k^2/16C_1\mu_1 + k\chi\left(\frac{1}{4}\frac{p_1}{m}+\frac{1}{16}\frac{q}{m^2}-p_3-p_5\right)\Big/2\mu_1C_1. \tag{2.60}$$

Therefore, the solution to second-order is

$$\eta = \epsilon\cos\theta + \epsilon^2 a_{22}(\cos 2\theta - \cos\theta_2) + O(\epsilon^3), \tag{2.61}$$

$$\varphi = \epsilon\mu_1 \sin\theta \frac{\cosh(y+h)}{\sinh h} + \epsilon^2\left\{\tfrac{1}{4}\mu_1^2(C_1^2-1)\,t + [\mu_1 b_{22}\cos 2\theta - \tfrac{1}{2}\mu_2 a_{22}\sin\theta_2]\frac{\cosh 2(y+h)}{\sinh 2h}\right\} + O(\epsilon^3), \tag{2.62}$$

$$\Phi = \epsilon m^{-1}\sin\theta\, e^{-my} + \epsilon^2\left\{\left[\tfrac{1}{8}(\gamma+1)\frac{M^4}{m^3}y + \frac{d_{22}}{m}\right]\sin 2\theta - \frac{a_{22}}{m}e^{-2my}\sin\theta_2\right\}e^{-2my} + O(\epsilon^3), \tag{2.63}$$

where

$$\theta = x + (\mu_1 + \epsilon^2\beta')\,t + O(\epsilon^3), \tag{2.64}$$

$$\theta_2 = 2x + \mu_2 t + O(\epsilon^2). \tag{2.65}$$

If the effects of the external gas are negligible (i.e $\chi = 0$), the above solution reduces to that of Nayfeh (1969). If, in addition, the body force, g, is directed from the gas to the liquid and if the terms involving θ_2 are eliminated, the above solution reduces to those of Barakat & Houston (1968) and Nayfeh (1970*a*).

For real μ_1, the above solution shows to second-order that an initially harmonic disturbance travels with two wave speeds; one is amplitude dependent and the other is amplitude independent as in the case of no external gas. However, the external gas affects the linear as well as the non-linear contribution to the wave speed. For imaginary μ_1, on the other hand, the gas/liquid interface grows without limit, and no travelling wave solutions are possible. An expansion for standing waves is presented in the next section.

Although the above solution is valid for a wide range of wave-numbers such that μ_1 is real, it breaks down as $\mu_1 \to 0$ or $\mu_2 \to 2\mu_1$. The case $\mu_1 = 0$ corresponds to the cut-off wave-number, and in this case, $\beta' \to \infty$ as $\mu_1 \to 0$. An expansion valid near the cut-off wave-numbers is presented in §2.3. The case $\mu_2 = 2\mu_1$ corresponds to the motion of the first and second harmonic with the same linear wave speed, and it is a special case of the triad resonance (Phillips 1960; Benney 1962; McGoldrick 1965; Simmons 1969). In this case, $a_{22} \to \infty$ as $\mu_2 \to 2\mu_1$. An expansion for periodic waves with $\mu_2 \approx 2\mu_1$ is presented in § 2.4.

2.2. *Expansion for standing waves*

In this case, the solution of the first-order problem is

$$\eta_1 = \cos\alpha\cos x, \tag{2.66}$$

$$\varphi_1 = \mu_1 \sin\alpha\cos x\,\frac{\cosh(y+h)}{\sinh h}, \tag{2.67}$$

$$\Phi_1 = m^{-1}\cos\alpha\sin x\, e^{-my}, \tag{2.68}$$

where

$$\alpha = \mu_1 T_0 + \xi(T_1, T_2).$$

Then, (2.21)–(2.24) become

$$m^2\frac{\partial^2\Phi_2}{\partial x^2} + \frac{\partial^2\Phi_2}{\partial y^2} = -\frac{1}{4}\frac{M^4}{m^2}(\gamma+1)(1+\cos 2\alpha)\sin 2x\, e^{-2my}, \tag{2.69}$$

$$\frac{\partial\eta_2}{\partial T_0}+\frac{\partial\varphi_2}{\partial y} = -\frac{\partial\xi}{\partial T_1}\sin\alpha\cos x-\tfrac{1}{2}\mu_1 C_1\sin 2\alpha\cos 2x, \tag{2.70}$$

$$\frac{\partial\eta_2}{\partial x}-\frac{\partial\Phi_2}{\partial y} = \frac{1}{4}\left(m+\frac{1}{m}\right)(1+\cos 2\alpha)\sin 2x, \tag{2.71}$$

$$\begin{aligned}-\eta_2-\frac{\partial\varphi_2}{\partial T_0}-k^2\frac{\partial^2\eta_2}{\partial x^2}-mk\chi\frac{\partial\Phi_2}{\partial x} &\\ = \mu_1 C_1\frac{\partial\xi}{\partial T_1}\cos\alpha\cos x+\tfrac{1}{8}(1-C_1^2)\mu_1^2+\tfrac{1}{8}(3+C_1^2)\mu_1^2\cos 2\alpha &\\ +\{\tfrac{1}{8}[(C_1^2+1)\mu_1^2-2mk\chi]+\tfrac{1}{8}[(3-C_1^2)\mu_1^2-2mk\chi]\cos 2\alpha\}\cos 2x.&\end{aligned} \tag{2.72}$$

For there to be no secular terms in η_2, $\partial\xi/\partial T_1 = 0$, or $\xi = \xi(T_2)$, and hence, the solution of the second-order problem is

$$\eta_2 = [\tfrac{1}{2}a_{22}(\cos 2\alpha-\cos\mu_2 T_0)+a_{20}(1-\cos\mu_2 T_0)]\cos 2x, \tag{2.73}$$

$$\begin{aligned}\varphi_2 = -\tfrac{1}{8}(1-C_1^2)\mu_1^2 T_0-\tfrac{1}{16}(3+C_1^2)\mu_1\sin 2\alpha &\\ +[\tfrac{1}{2}\mu_1 b_{22}\sin 2\alpha-\tfrac{1}{4}\mu_2 a_{22}\cos\mu_2 T_0]\cos 2x\frac{\cosh 2(y+h)}{\sinh 2h},&\end{aligned} \tag{2.74}$$

$$\begin{aligned}\Phi_2 = \Bigg[\frac{1}{16}\frac{M^4}{m^3}(\gamma+1)(1+\cos 2\alpha)\,y+\frac{1}{2m}(d_{22}\cos 2\alpha-a_{22}\cos\mu_2 T_0) &\\ +\frac{1}{m}(d_{20}-a_{20}\cos\mu_2 T_0)\Bigg]\sin 2x\,e^{-2my},&\end{aligned} \tag{2.75}$$

where

$$a_{20} = \left[2(1+C_1^2)\mu_1^2+\frac{k\chi}{m}\left(4+\frac{M^4}{m^2}(\gamma+1)\right)\right]\Big/8C_2\mu_2^2, \tag{2.76}$$

$$d_{20} = a_{20}+\frac{1}{8}\left(m+\frac{1}{m}\right)+\frac{1}{32}\frac{M^4}{m^3}(\gamma+1). \tag{2.77}$$

Substituting the first- and second-order solutions into (2.26)–(2.29), we get

$$m^2\frac{\partial^2\Phi_3}{\partial x^2}+\frac{\partial^2\Phi_3}{\partial y^2} = (P_1+Qy)\,e^{-3my}\cos\alpha\sin x+\text{NSPT}, \tag{2.78}$$

$$\frac{\partial\eta_3}{\partial T_0}+\frac{\partial\varphi_3}{\partial y} = (P_2+\xi')\sin\alpha\cos x+\text{NSPT}, \tag{2.79}$$

$$\frac{\partial\eta_3}{\partial x}-\frac{\partial\Phi_3}{\partial y} = P_3\cos\alpha\sin x+\text{NSPT}, \tag{2.80}$$

$$-\eta_3-\frac{\partial\varphi_3}{\partial T_0}-k^2\frac{\partial^2\eta_3}{\partial x^2}-mk\chi\frac{\partial\Phi_3}{\partial x} = (P_4+k\chi P_5+\mu_1 C_1\xi')\cos\alpha\cos x+\text{NSPT}, \tag{2.81}$$

where the P's and Q are defined in the appendix and $\xi' = d\xi/dT_2$. The condition which must be satisfied for there to be no secular terms is

$$\xi' = \left[-P_4-\mu_1 C_1 P_2+k\chi\left(\frac{1}{4}\frac{P_1}{m}+\frac{1}{16}\frac{Q}{m^2}-P_3-P_5\right)\right]\Big/2\mu_1 C_1. \tag{2.82}$$

The surface elevation to second-order is thus given by

$$\begin{aligned}\eta = \epsilon\cos(\mu_1+\epsilon^2\xi')t\cos x+\epsilon^2 &\\ \times\{\tfrac{1}{2}a_{22}[\cos 2(\mu_1+\epsilon^2\xi')t-\cos\mu_2 t]+a_{20}(1-\cos\mu_2 t)\}\cos 2x+O(\epsilon^3).&\end{aligned} \tag{2.83}$$

If the effects of the external gas are neglected, this solution reduces to that of Nayfeh (1969). For real μ_1 (2.83) represents oscillating standing waves with an amplitude dependent frequency given by $\mu_1+\epsilon^2\xi'$ and an amplitude independent frequency given by μ_2. If μ_1 is imaginary, (2.83) represents growing waves, and it is valid for short times only, because as time increases, the second term dominates the first term. Although this expansion is valid for a wide range of wave-numbers, it is singular at the cut-off wave-number (i.e. $\mu_1 \approx 0$), and at the second harmonic resonant wave-numbers (i.e $\mu_2 \approx 2\mu_1$) as in the travelling wave case. An expansion valid near the cut-off wave-numbers is presented in the next section.

2.3. *Expansion valid near the cut-off wave-number*

To determine an expansion valid near the cut-off wave-number, we let

$$k = k_c+\epsilon^2\sigma \quad \text{with} \quad \sigma = O(1). \tag{2.84}$$

Then, (2.16)–(2.30) remain unchanged except k is replaced by k_c and (2.29) is modified by the addition of the term $\sigma(2k_c[\partial^2\eta_1/\partial x^2]+m\chi[\partial\Phi_1/\partial x])$. In this case, we consider the variations with respect to the time scales T_0 and T_1 only so that the solution of the first-order problem becomes

$$\eta_1 = \eta_{11}(T_1)\cos x, \tag{2.85}$$

$$\varphi_1 = 0, \tag{2.86}$$

$$\Phi_1 = \epsilon m^{-1}\eta_{11}\sin x\, e^{-my}, \tag{2.87}$$

$$\eta_{11}(0) = 1, \quad \eta'_{11}(0) = 0, \tag{2.88}$$

where primes denote differentiation with respect to the arguments.

Then, (2.21)–(2.24) become

$$m^2\frac{\partial^2\Phi_2}{\partial x^2}+\frac{\partial^2\Phi_2}{\partial y^2} = -\tfrac{1}{2}(\gamma+1)\frac{M^4}{m^2}\eta_{11}^2\sin 2x\, e^{-2my}, \tag{2.89}$$

$$\frac{\partial\eta_2}{\partial T_0}+\frac{\partial\varphi_2}{\partial y} = -\eta'_{11}\cos x \quad \text{at} \quad y = 0, \tag{2.90}$$

$$\frac{\partial\eta_2}{\partial x}-\frac{\partial\Phi_2}{\partial y} = \frac{1}{2}\frac{1+m^2}{m}\eta_{11}^2\sin 2x \quad \text{at} \quad y = 0, \tag{2.91}$$

$$-\eta_2-\frac{\partial\varphi_2}{\partial T_0}-k_c^2\frac{\partial^2\eta_2}{\partial x^2}-mk_c\chi\frac{\partial\Phi_2}{\partial x} = -\tfrac{1}{2}mk_c\chi\eta_{11}^2\cos 2x \quad \text{at} \quad y = 0. \tag{2.92}$$

The solution of (2.1), (2.89)–(2.92) subject to the initial conditions (2.25) is

$$\eta_2 = a_{22}(\eta_{11}^2\cos 2x-\cos\theta_2), \tag{2.93}$$

$$\varphi_2 = -\eta'_{11}\cos x\frac{\cosh(y+h)}{\sinh h}-\tfrac{1}{2}\mu_2 a_{22}\sin\theta_2\frac{\cosh 2(y+h)}{\sinh 2h}, \tag{2.94}$$

$$\Phi_2 = \left[\tfrac{1}{8}(\gamma+1)\frac{M^4}{m^3}y+\frac{d_{22}}{m}\right]e^{-2my}\sin 2x-\frac{a_{22}}{m}e^{-2my}\sin\theta_2, \tag{2.95}$$

where a_{22} and d_{22} are evaluated at $k = k_c$.

Substituting for the first- and second-order solutions into (2.26)–(2.28) and the modified (2.29), we have

$$m^2\frac{\partial^2\Phi_3}{\partial x^2}+\frac{\partial^2\Phi_3}{\partial y^2} = (p_1+qy)\,\eta_{11}^3\,e^{-3my}\sin x+\text{NSPT}, \tag{2.96}$$

$$\frac{\partial\eta_3}{\partial T_0}+\frac{\partial\varphi_3}{\partial y} = 0+\text{NSPT}\quad\text{at}\quad y=0, \tag{2.97}$$

$$\frac{\partial\eta_3}{\partial x}-\frac{\partial\Phi_3}{\partial y} = p_3\eta_{11}^3\sin x+\text{NSPT}\quad\text{at}\quad y=0, \tag{2.98}$$

$$-\eta_3-\frac{\partial\varphi_3}{\partial T_0}-k_c\frac{\partial^2\eta_3}{\partial x^2}-mk_c\chi\frac{\partial\Phi_3}{\partial x}$$
$$= [(\tfrac{3}{8}k_c^2+k_c\chi p_5)\eta_{11}^3-C_1\eta_{11}''+(2k_c-\chi)\sigma\eta_{11}]\cos x+\text{NSPT}\quad\text{at}\quad y=0. \tag{2.99}$$

In these equations, the p's and q are evaluated at $k = k_c$.

The particular solution of the third-order problem contains secular terms which make η_3/η_1 unbounded as $T_0\to\infty$. The condition which must be satisfied for there to be no secular terms is

$$C_1\eta_{11}''+(2k_c-\chi)\sigma\eta_{11}-\Gamma\eta_{11}^3 = 0, \tag{2.100a}$$

where

$$\Gamma = \tfrac{3}{8}k_c^2+k_c\chi\left[p_3+p_5-\frac{1}{16}\frac{q}{m^2}-\frac{1}{4}\frac{p_1}{m}\right]. \tag{2.100b}$$

With (2.88), (2.100a) has the integral

$$\eta_1'^2 = (\Gamma/2C_1)(1-\eta_{11}^2)(\tilde{\Gamma}-\eta_{11}^2), \tag{2.101}$$

where

$$\tilde{\Gamma} = (2\sigma/\Gamma)(2k_c-\chi)-1.$$

Different cases arise depending on the signs and magnitudes of Γ and $\tilde{\Gamma}$. If $\Gamma > 0$, since $\eta_{11}(0) = 1$, η_{11}^2 cannot exceed 1 if $\tilde{\Gamma} > 1$ and cannot decrease from 1 if $\tilde{\Gamma} < 1$; otherwise, η_{11}' will be imaginary. Thus, if $\tilde{\Gamma} > 1$, η_{11} is bounded and oscillates between 1 and -1, and if $\tilde{\Gamma} < 1$, η_{11} is unbounded. The special case $\tilde{\Gamma} = 1$ separates stable from unstable disturbances, and hence,

$$\sigma = \Gamma/(2k_c-\chi) \tag{2.102a}$$

and

$$k = k_c+\epsilon^2\,\Gamma/(2k_c-\chi)+O(\epsilon^3) \tag{2.102b}$$

is the cut-off wave-number to second order. Since Γ is independent of the liquid depth, the cut-off wave-number is independent of the liquid depth as in the case of no external gas. On the other hand, if $\Gamma < 0$, (2.101) can be written as

$$\eta_{11}'^2 = -(\Gamma/2C_1)(1-\eta_{11}^2)(\eta_{11}^2-\tilde{\Gamma}). \tag{2.103}$$

In this case, η_{11}^2 is bounded and oscillates between 1 and $\tilde{\Gamma}$ if $\tilde{\Gamma}$ is positive, and oscillates between 0 and 1 if $\tilde{\Gamma}$ is negative. In all cases, the solution of (2.101) and (2.103) is given by Jacobian elliptic functions.

In the absence of an external gas, $\chi = 0$, and $k_c = 1$, and the cut-off wave-number (2.102) becomes

$$k = 1+\tfrac{3}{16}\epsilon^2+O(\epsilon^3) \tag{2.104}$$

in agreement with that obtained by Nayfeh (1969). This increase in the cut-off wave-number is due to the reduction of the surface tension force. The effect of the non-linear motion of the gas on the stability is represented by the term proportional to $k_c\chi$. As $M \to 1$, this term is positive while as $M \to 0$, this term may be positive or negative depending on the value of χ. Therefore, the non-linear motion of the gas may be stabilizing or destabilizing depending on the values of M and χ in contrast with the non-linear motion of the liquid which does not affect the stability to second order.

The second-order expansion for η which is valid in the region $k - k_c = O(\epsilon^2)$ is

$$\eta = \epsilon\eta_{11}(\epsilon t)\cos x + \tfrac{1}{2}\epsilon^2 \frac{k_c\chi}{2k_c^2+1}\left[\frac{1}{m} + \frac{1}{4}\frac{M^4}{m^3}(\gamma+1)\right][\eta_{11}^2(\epsilon t) - \cos\mu_2 t]\cos 2x + O(\epsilon^3), \tag{2.105}$$

where
$$\eta_{11} = cn\left[\left(\frac{(2k_c-\chi)\sigma - \Gamma}{C_1}\right)^{\frac{1}{2}}\epsilon t;\quad -\frac{1}{2}\frac{\Gamma}{(2k_c-\chi)\sigma-\Gamma}\right]. \tag{2.106}$$

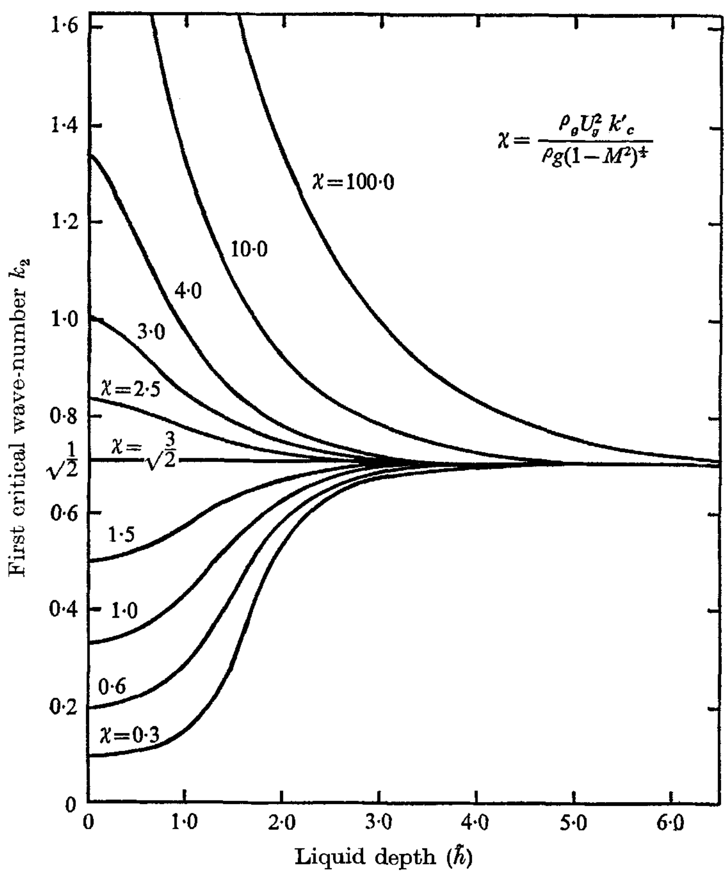

FIGURE 1. First critical wave-number, k_2, as a function of the liquid depth, $\bar{h}$, for different pressure perturbations.

2.4. *Periodic waves near second harmonic resonance*

This case occurs when the body force is directed from the gas to the liquid only. The wave-number k_2 for second harmonic resonance is the solution of $\mu_2 = 2\mu_1$ or

$$(4k^2 - 2k\chi + 1)\tanh 2k\bar{h} = (k^2 - k\chi + 1)\tanh k\bar{h}, \tag{2.107}$$

where $\tilde{h}$ is the dimensionless liquid depth with respect to k_c'. The solution of (2.107) is shown in figure 1 as a function of $\tilde{h}$ and χ. As $\tilde{h} \to \infty$, $k_2^2 = \frac{1}{2}$ for all χ. If $\chi = 3/\sqrt{2}$, $k_2^2 = \frac{1}{2}$ for all $\tilde{h}$. For small $\tilde{h}$, $k_2 = \frac{1}{3}\chi$. We determine next an expansion for periodic waves near k_2 when μ_1^2 and μ_2^2 are positive. This condition restricts the values of χ to those less than $3/\sqrt{2}$.

We assume that
$$k = k_2 + \epsilon\tilde{k} \tag{2.108}$$
with $\tilde{k} = O(1)$. This assumption leaves (2.16)–(2.24) unchanged except for the addition of the term
$$2k_2\tilde{k}\frac{\partial^2\eta_1}{\partial x^2} + mk_2\chi\frac{\partial\Phi_1}{\partial x}$$
to the right-hand side of (2.24). The solution of the first-order problem is taken to be
$$\eta_1 = \cos\theta + b\cos 2\theta, \tag{2.109}$$
$$\varphi_1 = \mu_1\sin\theta\frac{\cosh(y+h)}{\sinh h} + \mu_1 b\sin 2\theta\frac{\cosh 2(y+h)}{\sinh 2h}, \tag{2.110}$$
$$\Phi_1 = m^{-1}\sin\theta\, e^{-my} + m^{-1}b\sin 2\theta\, e^{-2my}, \tag{2.111}$$
$$\theta = x + \mu_1 T_0 + \beta_1(T_1). \tag{2.112}$$

Substituting this solution into (2.21)–(2.23), and the modified (2.24), we get
$$m^2\frac{\partial^2\Phi_2}{\partial x^2} + \frac{\partial^2\Phi_2}{\partial y^2} = -\tfrac{1}{2}(\gamma+1)\frac{M^4}{m^2}\sin 2\theta\, e^{-2my} - M^2\left(\frac{\gamma+1}{m^2} - \gamma + 3\right)b\sin\theta\, e^{-3my} + \text{NSPT}, \tag{2.113}$$
$$\frac{\partial\eta_2}{\partial T_0} + \frac{\partial\varphi_2}{\partial y} = [-\tfrac{1}{2}\mu_1(C_1 + 2C_2)b + \beta']\sin\theta - [\mu_1 C_1 - 2b\beta']\sin 2\theta + \text{NSPT} \quad \text{at} \quad y = 0, \tag{2.114}$$
$$\frac{\partial\eta_2}{\partial x} - \frac{\partial\Phi_2}{\partial y} = \frac{1}{2}\left(m + \frac{1}{m}\right)\sin 2\theta + \tfrac{3}{2}mb\sin\theta + \text{NSPT} \quad \text{at} \quad y = 0, \tag{2.115}$$
$$\begin{aligned}\eta_2 - \frac{\partial\varphi_2}{\partial T_0} - k_2^2\frac{\partial^2\eta_2}{\partial x^2} - mk_2\chi\frac{\partial\Phi_2}{\partial x} &= [\tfrac{1}{4}(3 - C_1^2)\mu_1^2 - \tfrac{1}{2}mk_2\chi - 8k_2\tilde{k} + 2\tilde{k}\chi b + 2\mu_1 C_2 b\beta']\cos 2\theta \\ &\quad + [\tfrac{1}{2}(3 - 2C_1C_2)\mu_1^2 b - \tfrac{1}{2}mk_2\chi b - 2k_2\tilde{k} + \tilde{k}\chi + \mu_1 C_1\beta']\cos\theta \\ &\quad + \text{NSPT} \quad \text{at} \quad y = 0,\end{aligned} \tag{2.116}$$
where $\beta' = d\beta/dT_1$.

To determine the conditions which must be satisfied for there to be no secular terms, we assume that the particular solution corresponding to the secular producing terms in (2.113)–(2.116) is
$$\eta_2 = 0, \tag{2.117}$$
$$\varphi_2 = A_1\sin\theta\frac{\cosh(y+h)}{\sinh h} + A_2\sin 2\theta\frac{\cosh 2(y+h)}{\sinh 2h}, \tag{2.118}$$
$$\Phi_2 = \tfrac{1}{8}(\gamma+1)\frac{M^4}{m^3}y\sin 2\theta\, e^{-2my} - \frac{1}{8}\frac{M^2}{m^2}\left(\frac{\gamma+1}{m^2} + \gamma + 3\right)b\sin\theta\, e^{-3my} + m^{-1}[D_1\sin\theta\, e^{-my} + D_2\sin 2\theta\, e^{-2my}]. \tag{2.119}$$

This solution satisfies (2.1) and (2.113). Substituting this solution into (2.113)–(2.116), and equating the coefficients of each harmonic on both sides, we get

$$A_1 = -\tfrac{1}{2}\mu_1(C_1+2C_2)\,b+\beta', \tag{2.120}$$

$$2A_2 = -\mu_1 C_1 + 2b\beta', \tag{2.121}$$

$$D_1 - \frac{3}{8}\frac{M^2}{m}\left(\frac{\gamma+1}{m^2}-\gamma+3\right)b = \tfrac{3}{2}mb, \tag{2.122}$$

$$2D_2 - \tfrac{1}{8}(\gamma+1)\frac{M^4}{m^3} = \frac{1}{2}\left(m+\frac{1}{m}\right), \tag{2.123}$$

$$-\mu_1 A_1 C_1 - k_2\chi D_1 + \tfrac{1}{8}k_2\chi\frac{M^2}{m}\left(\frac{\gamma+1}{m^2}-\gamma+3\right)b$$
$$= \tfrac{1}{2}(3-2C_1C_2)\,\mu_1^2 - \tfrac{1}{2}mk_2\chi - 2k_2\tilde{k} + k_2\chi + \mu_1\beta' C_1, \tag{2.124}$$

$$-2\mu_1 A_2 C_2 - 2k_2\chi D_2 = \tfrac{1}{4}(3-C_1^2)\,\mu_1^2 - \tfrac{1}{2}mk_2\chi - 8k_2\tilde{k} + 2k_2\chi b + 2\mu_1 b\beta' C_2. \tag{2.125}$$

Elimination of A_m and D_m from (2.120) to (2.125) yields

$$\beta' = R_1 b + \frac{2k_2-\chi}{2\mu_1 C_1}\tilde{k}, \tag{2.126}$$

$$b\beta' = R_2 + \frac{4k_2-\chi}{2\mu_1 C_1}\tilde{k}b, \tag{2.127}$$

where
$$4\mu_1 C_1 R_1 = C_1^2 + 4C_1C_2 - 3 - k_2\chi\left[\frac{1}{2}\frac{M^2}{m}\left(\frac{\gamma+1}{m^2}-\gamma+3\right)+2m\right], \tag{2.128}$$

$$16\mu_1 C_2 R_2 = C_1^2 + 4C_1C_2 - 3 - k_2\chi\left[\tfrac{1}{2}(\gamma+1)\frac{M^4}{m^3}+\frac{2}{m}\right]. \tag{2.129}$$

The solution of (2.126) and (2.127) for b is

$$b = \left(\frac{\sigma}{4} \pm \left(\frac{\sigma^2}{16}+R_1R_2\right)^{\frac{1}{2}}\right)\Big/ R_1, \tag{2.130}$$

where σ is the detuning and given by

$$\sigma = (\mu_2 - 2\mu_1)/\epsilon = \left(\frac{4k_2-\chi}{C_2} - \frac{2k_2-\chi}{C_1}\right)\frac{\tilde{k}}{\mu_1}. \tag{2.131}$$

In dimensional quantities, the surface elevation to first approximation is given by

$$\eta' = a\cos k'(x'+ct') + ba\cos 2k'(x'+ct') + O(a^2), \tag{2.132}$$

where the wave speed c is given by

$$c = \frac{Tg^{\frac{1}{4}}}{\rho k^2}\left\{\mu_1 + ak'\left[\frac{\sigma}{4} + \frac{2k_2-\chi}{2\mu_1 C_1}\tilde{k} - \frac{\tilde{k}\mu_1}{2k_2} \mp \left(\frac{\sigma^2}{16}+R_1R_2\right)^{\frac{1}{2}}\right]\right\} + O(a^2). \tag{2.133}$$

This solution shows that for periodic waves, the amplitude of the second harmonic is not arbitrary, but it is a function of the amplitude of the first harmonic, the deviation from k_2, the liquid depth, and the pressure perturbation exerted by the gas on the liquid/gas interface.

In the absence of an external gas (i.e. $\chi = 0$), the above solution reduces to that of Barakat & Houston (1968), and to the first-order solution of Nayfeh (1970*a*). As $h \to \infty$, these solutions reduce, in turn, to those of Pierson & Fife

(1961) and Wilton (1915) with $\sigma = 0$. These solutions predict double-dimpled wave profiles. Schooley (1960) observed double-dimpled wave profiles by means of enlarged pictures of short-fetch, wind-generated waves.

Subsequent to the submittal of this paper, Nayfeh (1970*b*) obtained a first-order expansion at or near k_2 for the spatial as well as temporal variation of the amplitudes and the phases of the fundamental and the second harmonic. The results show that the above calculated periodic wave profiles are unstable for certain gas flow conditions, and completely stable in the absence of the external gas. This last expansion of Nayfeh generalizes those of Simons (1969) and McGoldrick (1970) by including the effects of (1) the external gas, (2) near resonance, and (3) liquid depth.

3. Case of viscous liquid and inviscid subsonic or supersonic gas

In this case, we investigate the effect of liquid viscosity on the stability of an initially quiscent liquid layer adjacent to an inviscid subsonic or supersonic gas. To describe the motion of the liquid, we introduce the characteristic velocity $u_0 = (gh_0)^{\frac{1}{2}}$ and time $\tau = (k'u_0)^{-1}$, where h_0 is the dimensional depth of the undisturbed liquid layer and k' is a characteristic wave-number of the disturbance. If primes denote dimensional quantities, we introduce the following dimensionless quantities

$$x = x'k', \quad y = y'/h_0, \quad h = h'/h_0,$$
$$u = u'/u_0, \quad v = v'/u_0\alpha, \quad \alpha = k'h_0,$$
$$t = t'/\tau, \quad p = [p' + g(y' - h_0) - p_0']/\rho u_0^2,$$

where h' is the depth of the disturbed liquid layer, p is the liquid pressure, and u and v are the liquid velocities in the x and y directions, respectively. Here, the x and y axes are in and normal to the solid surface rather than in the interface as treated above.

In terms of these dimensionless quantities, the liquid motion is governed by

$$u_x + v_y = 0, \tag{3.1}$$
$$u_t + uu_x + vu_y = -p_x + (\alpha R)^{-1}(\alpha^2 u_{xx} + u_{yy}), \tag{3.2}$$
$$v_t + uv_x + vv_y = -\alpha^{-2}p_y + (\alpha R)^{-1}(\alpha^2 v_{xx} + v_{yy}), \tag{3.3}$$

where
$$R = u_0 h_0/\nu, \tag{3.4}$$

with ν the liquid kinematic viscosity. Equation (3.1) is satisfied if we introduce the stream function $\psi(x, y, t)$ defined by

$$u = \psi_y, \quad v = -\psi_x. \tag{3.5}$$

Eliminating p from (3.2) and (3.3), using (3.5), and rearranging, yield

$$\psi_{yyyy} = \alpha R(\psi_{yyt} + \psi_y\psi_{xyy} - \psi_{yyy}\psi_x) - 2\alpha^2\psi_{xxyy} + \alpha^3 R(\psi_{xxt} + \psi_y\psi_{xxx} - \psi_x\psi_{xxy}) - \alpha^4\psi_{xxxx}. \tag{3.6}$$

Equation (3.6) is supplemented by five boundary conditions: two at the solid/liquid interface and three at the gas/liquid interface. At the solid/liquid interface, both components of velocity vanish; that is,

$$\psi_x(x, 0, t) = \psi_y(x, 0, t) = 0. \tag{3.7}$$

Since the gas/liquid interface moves with the liquid velocity,

$$h_t + \psi_y h_x + \psi_x = 0, \quad y = h. \tag{3.8}$$

The balance of stresses at the gas/liquid interface gives

$$\psi_{yy} = \alpha^2(\psi_{xx} + h_x^2 \psi_{yy} + 4\psi_{xy} h_x) - \alpha^4 \psi_{xx} h_x^2, \quad y = h, \tag{3.9}$$

$$-p - (h-1) - \frac{2\alpha}{R}\psi_{xy}\frac{1-\alpha^2 h_x^2}{1+\alpha^2 h_x^2} - \frac{k^2 h_{xx}}{(1+\alpha^2 h_x^2)^{\frac{3}{2}}} + \tfrac{1}{2}mk\chi\alpha^{-1}C_p$$
$$- \frac{2\alpha}{R}(\psi_{yy} - \alpha^2\psi_{xx})\frac{h_x}{1+\alpha^2 h_x^2} = 0, \quad y = h, \tag{3.10}$$

where
$$p_x = (\alpha R)^{-1}\psi_{yyy} - (\psi_{yt} + \psi_y \psi_{xy} - \psi_{yy}\psi_x) + \alpha R^{-1}\psi_{xxy}, \tag{3.11}$$

$$k^2 = \frac{Tk'^2}{\rho g} = \alpha^2 \frac{T}{\rho g h_0^2}, \quad m = (|1-M^2|)^{\frac{1}{2}}, \quad \chi = \frac{\rho_g U_g^2}{m(\rho g T)^{\frac{1}{2}}} \tag{3.12}$$

and C_p is the subsonic or supersonic pressure perturbation coefficient exerted by the gas on the liquid/gas interface due to the appearance of waves and given by (2.8).

The gas motion is represented by the potential function $U_g[x + \Phi(x, \tilde{y}, t)]/k'$ where $\tilde{y} = y'k'$ and Φ satisfies (2.2) and (2.4) with y replaced by $\tilde{y}$. The boundary condition (2.6) becomes

$$\alpha h_x - \Phi_{\tilde{y}} = -\alpha h_x \Phi_x \quad \text{at} \quad \tilde{y} = \alpha(h-1). \tag{3.13}$$

In § 3.1, an expansion is obtained for long waves and small but finite amplitudes. This expansion is then specialized to the case of a sinusoidal initial disturbance in § 3.2.

3.1. *Solution for long waves*

An approximate solution for (2.2), (2.4), and (3.1)–(3.13) is sought for small α. Thus, we assume that

$$\psi = \alpha\psi_1 + \alpha^2\psi_2 + \alpha^3\psi_3 + \ldots, \tag{3.14}$$

$$p = p_0 + \alpha p_1 + \alpha^2 p_2 + \ldots, \tag{3.15}$$

$$\Phi = \alpha\Phi_1 + \alpha^2\Phi_2 + \alpha^3\Phi_3 + \ldots. \tag{3.16}$$

Let us discuss the solution for Φ first, in order to determine C_p, and then determine ψ and p. Substituting (3.16) into (2.2), (2.4), and (3.13), we get

Order α:

$$m^2\Phi_{1xx} + \Phi_{1\tilde{y}\tilde{y}} = 0, \tag{3.17}$$

$$\Phi_1 \to 0 \quad \text{as} \quad \tilde{y} \to \infty \quad \text{for} \quad m^2 > 0, \tag{3.18}$$

$$\Phi_{1\tilde{y}} = h_x \quad \text{at} \quad \tilde{y} = 0. \tag{3.19}$$

Order α^2:

$$m^2\Phi_{2xx} + \Phi_{2\tilde{y}\tilde{y}} = M^2[(\gamma+1)\Phi_{1x}\Phi_{1xx} + (\gamma-1)\Phi_{1x}\Phi_{1\tilde{y}\tilde{y}} + 2\Phi_{1\tilde{y}}\Phi_{1\tilde{y}\tilde{y}}], \tag{3.20}$$

$$\Phi_2 \to 0 \quad \text{as} \quad \tilde{y} \to \infty \quad \text{for} \quad m^2 > 0, \tag{3.21}$$

$$\Phi_{2\tilde{y}} = h_x\Phi_{1x} - \Phi_{1\tilde{y}\tilde{y}}(h-1) \quad \text{at} \quad \tilde{y} = 0. \tag{3.22}$$

We will not write down the problem for Φ_3; it is sufficient to observe that Φ_3 is a non-linear function of $(h-1)$ and its derivatives, and hence, it will not contribute to the expansion to $O[(h-1)^3] = O(\alpha^3)$, which we will obtain later. The pressure coefficient is then

$$C_p = -2\alpha\Phi_{1x} - \alpha^2(2\Phi_{2x} + m^2\Phi_{1x}^2 + \Phi_{1y}^2) + \alpha^3 \quad \text{(non-linear terms)}. \quad (3.23)$$

We now turn to ψ and p. Since (3.3) is linear, it is satisfied by each ψ_i. Substituting (3.14) and (3.15) into (3.6)–(3.11) and (3.23), equating coefficients of equal powers of α on both sides, and assuming $R = 0(1)$, we obtain the following equations.

Order α:

$$\psi_{1yyyy} = 0, \quad (3.24)$$

$$\psi_{1yy} = 0, \quad y = h, \quad (3.25)$$

$$p_0 + (h-1) + k^2h_{xx} - mk\chi\,\Phi_{1x} = 0, \quad y = h \quad (3.26a)$$

and

$$p_{0x} = (1/R)\,\psi_{1yyy}. \quad (3.26b)$$

Order α^2:

$$\psi_{2yyyy} = R\psi_{1yyt}, \quad (3.27)$$

$$\psi_{2yy} = 0, \quad y = h, \quad (3.28)$$

$$p_1 = -\tfrac{1}{2}mk\chi[2\Phi_{2x} + m^2\Phi_{1x}^2 + \Phi_{1y}^2], \quad y = h, \quad (3.29a)$$

and

$$p_{1x} = (1/R)\,\psi_{2yyy} - \psi_{1yt}. \quad (3.29b)$$

Order α^3:

$$\psi_{3yyyy} = R(\psi_{2yyt} + \psi_{1y}\psi_{1xyy} - \psi_{1yyy}\psi_{1x}) - 2\psi_{1xxyy}, \quad (3.30)$$

$$\psi_{3yy} = \psi_{1xx} + h_x^2\psi_{1yy} + 4\psi_{1xy}h_x, \quad y = h, \quad (3.31)$$

$$p_2 + 2R^{-1}(\psi_{1xy} + \psi_{1yy}h_x) - \tfrac{3}{2}k^2h_x^2h_{xx} + \text{non-linear terms in } \Phi = 0, \quad y = h, \quad (3.32a)$$

$$p_{2x} = R^{-1}(\psi_{3yyy} + \psi_{1xxy}) - \psi_{2yt} - \psi_{1y}\psi_{1xy} + \psi_{1yy}\psi_{1x}. \quad (3.32b)$$

The solution of the first-order problem is

$$\psi_1 = a_2y^2 + a_3y^3, \quad (3.33)$$

where

$$a_2 = \tfrac{1}{2}Rh\tilde{\zeta} \quad \text{and} \quad a_3 = -\tfrac{1}{6}R\tilde{\zeta}, \quad (3.34)$$

with

$$\tilde{\zeta} = h_x + k^2h_{xxx} - mk\chi\,\Phi_{1xx}. \quad (3.35)$$

On using the above solution and solving the second-order problem, we get

$$\psi_2 = -\tfrac{1}{2}R(p_{1x}h + a_{2t}h^2 + a_{3t}h^3)\,y^2 + \tfrac{1}{6}Rp_{1x}y^3 + R(\tfrac{1}{12}a_{2t}y^4 + \tfrac{1}{20}a_{3t}y^5). \quad (3.36)$$

Then, the solution of the third-order problem is

$$\psi_3 = b_2y^2 + b_3y^3 - \tfrac{1}{12}R(\tilde{\zeta}_{xx} + \tfrac{1}{6}R^2\tilde{\zeta}_{tt})\,y^4 + \tfrac{1}{60}R\tilde{\zeta}_{xx}y^5 + \tfrac{1}{720}R^3\tilde{\zeta}_{tt}y^6 - \tfrac{1}{5040}R^3\tilde{\zeta}_{tt}y^7 + \text{non-linear terms}, \quad (3.37)$$

where

$$b_2 = \tfrac{3}{4}R\tilde{\zeta}_{xx} - \tfrac{1}{15}R^3\tilde{\zeta}_{tt} \quad \text{and} \quad b_3 = -\tfrac{1}{12}R\tilde{\zeta}_{xx}. \quad (3.38)$$

So far, no restriction has been imposed on the magnitude of the disturbance. In order to get an approximate solution to the liquid motion, the disturbance amplitude is assumed to be small but finite. Thus, we assume that

$$h = 1+\epsilon\eta(x,t), \tag{3.39}$$

where ϵ is a small but finite dimensionless constant. Substituting the above expressions for ψ_0, ψ_1, ψ_2, and h into (3.8), assuming that $\epsilon = O(\alpha)$, and keeping up to third-order quantities, we obtain the following equation:

$$\eta_t + \tfrac{1}{3}\alpha R\zeta_x - \tfrac{2}{15}\alpha^2R^2\zeta_{xt} + \tfrac{3}{5}\alpha^3R\zeta_{xxx} + \epsilon\alpha R(\eta\zeta_x+\zeta\eta_x) + \epsilon^2\alpha R\eta(\eta\zeta_x + 2\zeta\eta_x)$$
$$+\tfrac{1}{6}\alpha^2\epsilon^{-1}R\,mk\chi\frac{\partial}{\partial x}[2\Phi_{2x} + m^2\Phi_{1x}^2 + \Phi_{1y}^2] + O(\epsilon^m\alpha^n) = 0, \tag{3.40}$$

where $m+n = 4$ and
$$\zeta = \eta_x + k^2\eta_{xxx} - mk\chi\,\epsilon^{-1}\,\Phi_{1xx}. \tag{3.41}$$

3.2. *Linear case*

In this case, we let $\epsilon = 0$ in (3.31) and obtain

$$\eta_t + \tfrac{1}{3}\alpha R\zeta_x - \tfrac{2}{15}\alpha^2R^2\zeta_{xt} + \tfrac{3}{5}\alpha^3R\zeta_{xxx} + O(\alpha^4) = 0. \tag{3.42}$$

We assume that
$$\eta = \exp(ix+\mu_1 t) \tag{3.43}$$

and substitute into (3.42), (3.17)–(3.19), and (3.41) to determine Φ_1 and μ_1.

When the gas flow is subsonic, that is, $m^2 > 0$, the solution for Φ_1 is given as

$$\Phi_1 = -i(\epsilon/m)\,e^{-m\tilde{y}}\,\eta, \tag{3.44}$$

while C_p and ζ are given as

$$C_p = -2\alpha(\epsilon/m)\,\eta, \quad \zeta = -i(k^2-\chi k-1)\,\eta, \tag{3.45}$$

indicating that the pressure perturbation is in antiphase with wave amplitude. In this case with the body force directed from the liquid to the gas, we obtain

$$\mu_1 = -\tfrac{1}{3}\alpha R(k^2-\chi k-1)\{1-\tfrac{9}{5}\alpha^2[1-\tfrac{2}{27}R^2(k^2-\chi k-1)]\} + O(\alpha^4). \tag{3.46a}$$

If the body force is directed from the gas to the liquid, (3.46a) is modified to read

$$\mu_1 = -\tfrac{1}{3}\alpha R(k^2-\chi k+1)\{1-\tfrac{9}{5}\alpha^2[1-\tfrac{2}{27}R^2(k^2-\chi k+1)]\} + O(\alpha^4). \tag{3.46b}$$

Equations (3.46) show that viscosity did not alter the cut-off wave-numbers given by (2.36). As in the inviscid case, the cut-off wave-number is independent of the liquid depth; however, decreasing the liquid depth decreases the amplification rate. The Kelvin–Helmholtz criterion for complete stabilization in (3.46b), i.e. $\chi \leqslant 2$, still holds in the very viscous case, since the second term inside the braces is always much less than one.

When the gas flow is supersonic, that is, $M > 1$, the solution for Φ_1 is given as

$$\Phi_1 = -(\epsilon/m)\,e^{-im\tilde{y}}\,\eta, \tag{3.47}$$

$$C_p = 2\alpha(\epsilon/m)\,\eta_x, \quad \zeta = -i(k^2+i\chi k-1)\,\eta, \tag{3.48}$$

where only the right-running characteristics have meaning and the pressure perturbation is in phase with the wave slope. In this case, we find

$$\mu_1 = -\tfrac{1}{3}\alpha R(k^2+i\chi k-1)\{1-\tfrac{9}{5}\alpha^2[1-\tfrac{2}{27}R^2(k^2+i\chi k-1)]\} + O(\alpha^4). \tag{3.49}$$

The real and imaginary parts of (3.49) for the case of the body force directed from the liquid to the gas are

$$\mu_{1r} = \tfrac{1}{3}\alpha R(k^2-1)\{1-\tfrac{9}{5}\alpha^2[1-\tfrac{2}{27}R^2(k^2-1)]\}+\tfrac{2}{45}\alpha^3R^3\chi^2k^2+O(\alpha^4), \quad (3.50a)$$

$$\mu_{1i} = -\tfrac{1}{3}\alpha R\chi k\{1-\tfrac{9}{5}\alpha^2[1-\tfrac{2}{27}R^2(k^2-1)]\}-\tfrac{2}{45}\alpha^3R^3\chi k(k^2-1)+O(\alpha^4). \quad (3.50b)$$

The cut-off wave-number from (3.41 a) is given by

$$k_c = 1+\tfrac{1}{15}\alpha^2R^2\chi^2+O(\alpha^3). \quad (3.51)$$

In the inviscid case (Chang & Russell 1965), no cut-off wave-number exists with the supersonic flow, regardless of the direction of the body force. An increase in viscosity or Mach number decreases the cut-off wave-number, which is stabilizing.

For the case of the body force directed from the gas to the liquid, the growth rate and oscillation frequency are

$$\mu_{1r} = -\tfrac{1}{3}\alpha R(k^2+1)\{1-\tfrac{9}{5}\alpha^2[1-\tfrac{2}{27}R^2(k^2+1)]\}+\tfrac{2}{45}\alpha^3R^3\chi^2k^2+O(\alpha^4), \quad (3.52a)$$

$$\mu_{1i} = -\tfrac{1}{3}\alpha R\chi k\{1-\tfrac{9}{5}\alpha^2[1-\tfrac{2}{27}R^2(k^2+1)]\}-\tfrac{2}{45}\alpha^3R^3\chi k(k^2+1)+O(\alpha^4), \quad (3.52b)$$

and $\mu_{1r} < 0$ for all k, indicating complete stabilization with large viscosity.

3.3. *Non-linear case with subsonic external flow*

By letting $$\eta = \eta_1(t)\,e^{ix}+\bar{\eta}_1(t)\,e^{-ix}+\epsilon[\eta_2(t)\,e^{2ix}+\bar{\eta}_2(t)\,e^{-2ix}]+O(\epsilon^2) \quad (3.53)$$

in (3.40) and (3.41), using (3.17)–(3.22), and noting that the solution for Φ_2 produces a higher harmonic such that neither Φ_1 nor Φ_2 will contribute to the last term in (3.40) for the solution of η_1, so that (3.45) is still valid, we find that

$$d\eta_1/dt-\mu_1\eta_1 = -\epsilon^2\alpha R[(7k^2-3\chi k-1)\,\bar{\eta}_1\eta_2+(k^2-\chi k-1)\,\eta_1^2\bar{\eta}_1]+O(\epsilon^3), \quad (3.54)$$

$$d\eta_2/dt+\tfrac{4}{3}\alpha R(4k^2-2\chi k-1)\,\eta_2 = -2\alpha R(k^2-\chi k-1)\,\eta_1^2+O(\epsilon), \quad (3.55)$$

where μ_1 is given by (3.46) and the body force is directed from the liquid to the gas. To this order, the non-linear motion of the gas does not affect the motion of the liquid. As a first approximation,

$$\eta_1 = c\,e^{-\frac{1}{3}\alpha R(k^2-\chi k-1)\,t}, \quad (3.56)$$

where c is a constant. Then, from (3.55),

$$\eta_2 = -3\,\frac{k^2-\chi k-1}{7k^2-3\chi k-1}\,\eta_1^2. \quad (3.57)$$

Substituting for η_2 from (3.57) into (3.54) gives

$$d\eta_1/dt = -\tfrac{1}{3}\alpha R(k^2-\chi k-1)\{1-\tfrac{9}{5}\alpha^2[1-\tfrac{2}{27}R^2(k^2-\chi k-1)]-6\epsilon^2\eta_1^2\}\,\eta_1. \quad (3.58)$$

Equation (3.58) shows that the non-linearity did not alter the cut-off wave-number, and it is still given by (2.36), whereas it is amplitude dependent in the inviscid case. However, the rate of growth of unstable modes and the rate of decay of stable modes decrease as amplitude increases. If $\chi = 0$, the problem reduces to the non-linear Rayleigh–Taylor instability with large viscosity. If the body force is directed from the gas to the liquid, the term $(k^2-\chi k-1)$ is replaced by $(k^2-\chi k+1)$.

3.4. *Non-linear case with supersonic external flow*

Substituting (3.53) into (3.40) and (3.41) and noting that Φ_2 does not contribute to η_1 so that (3.48) is still valid, we get

$$d\eta_1/dt - \mu_1\eta_1 = -\epsilon^2\alpha R[(7k^2+5i\chi k-1)\bar{\eta}_1\eta_2 + (k^2+3i\chi k-1)\eta_1^2\bar{\eta}_1] + O(\epsilon^4), \quad (3.59)$$

$$d\eta_2/dt + \tfrac{4}{3}\alpha R(4k^2+2i\chi k-1)\eta_2 = -2\alpha R(k^2+i\chi k-1)\eta_1^2 + O(\epsilon^2), \quad (3.60)$$

where μ_1 is given by (3.49) and the body force is directed from the liquid to the gas. As a first approximation,

$$\eta_1 = c\,e^{-\frac{1}{3}\alpha R(k^2+i\chi k-1)t}. \quad (3.61)$$

Then, from (3.60),
$$\eta_2 = -3\frac{k^2+i\chi k-1}{7k^2+3i\chi k-1}\eta_1^2. \quad (3.62)$$

Substituting for η_2 from (3.62) into (3.59) gives

$$\frac{d\eta_1}{dt} = \mu_1\eta_1 + 2\frac{(7k^2-1)(k^2-1)-24\chi^2k^2+12i(k^2-1)\chi k}{7k^2+3i\chi k-1}\epsilon^2\alpha R\eta_1^2\bar{\eta}_1. \quad (3.63)$$

Letting $\eta_1 = a(t)\,e^{i\theta(t)}$, where a and θ are real in (3.63) .and separating real and imaginary parts, we get

$$\frac{da}{dt} = -\tfrac{1}{3}\alpha R\{(k^2-1)[1-\tfrac{9}{5}\alpha^2+\tfrac{2}{15}\alpha^2R^2(k^2-1)] - \tfrac{2}{45}\alpha^2R^2\chi^2k^2\}a$$
$$+\tfrac{1}{3}\alpha R\left\{6\frac{(7k^2-1)[(7k^2-1)(k^2-1)-24\chi^2k^2]-35(k^2-1)\chi^2}{(7k^2-1)^2+9\chi^2k^2}\right\}\epsilon^2a^3. \quad (3.64)$$

From (3.64), the cut-off wave-number is approximately given by

$$k_c = 1 + \tfrac{1}{15}\alpha^2R^2\chi^2 - 48\frac{\chi^2}{\chi^2+4}\epsilon^2a^2 + O(\alpha^m\epsilon^n), \quad n+m=3. \quad (3.65)$$

Thus, in this case also, the non-linearity is stabilizing. However, the cut-off wave-number is amplitude dependent in the supersonic case, while it is amplitude independent in the subsonic case. When the body force is directed from the gas to the liquid, the non-linearity does not change the linear result of complete stabilization. For wave-numbers near the cut-off wave-number for $\mu_1 > 0$, say, $k-1 = O(\alpha^m\epsilon^n)$ for $m+n=2$, disturbances do not grow indefinitely with time. In fact, at a given wave-number, a steady-state amplitude can be calculated from (3.65).

4. Concluding remarks

The results of § 2 show that, for an inviscid liquid and a subsonic flow, the cut-off wave-numbers are amplitude dependent, contrary to the linearized theory of Chang & Russell (1965) and Willson & Chang (1967). The cut-off wave-numbers are given by (2.102). The non-linear motion of the gas may increase or decrease the cut-off wave-number while the non-linear motion of the liquid does not affect this cut-off wave-number. Thus, the non-linear motion of the gas may be stabilizing or destabilizing. If the body force (g) is directed from the liquid to the gas, only one cut-off wave-number exists. Above this cut-off wave-number, standing as well as travelling waves are stable and have amplitude-dependent frequencies and wave velocities, respectively. If g is directed toward the liquid, then, depending on the ratio of the pressure perturbation to g, there are either two or no cut-off wave-numbers. In the former case, only disturbances with wave-numbers between the two cut-off wave-numbers are unstable. In the latter case, all disturbances are stable.

In the case with no cut-off wave-numbers, the expansions break down at a denumerable set of critical wave-numbers as in the case of negligible external gas effects (Pierson & Fife 1961; Barakat & Houston 1968; Nayfeh 1970*a*). An expansion, valid near the first critical wave-number (corresponding to a wave-length of 2·44 cm in deep water), is presented for periodic waves. It shows that two possible wave profiles could exist at or near this critical wave-number. One is gravity-like, with a wave velocity increasing with increasing amplitude; the second is capillary-like with a wave velocity decreasing with increasing amplitude. The analysis of Nayfeh (1970*b*) shows that these wave profiles may be unstable depending on the gas flow conditions.

It should be noted that travelling waves resemble standing waves near the cut-off wave-numbers. Moreover, the non-linear travelling waves (2.16) cannot be obtained by the superposition of the non-linear standing waves (2.27) as in the linear case.

For liquid Reynolds number $= O(1)$ and for a subsonic external flow, the linear cut-off wave-number is independent of viscosity. The non-linear cut-off wave-number is independent of the amplitude, in contrast with the inviscid liquid case. Moreover, the non-linear motion of the gas has no effect, whereas the non-linear motion of the liquid is stabilizing in the viscous case, and it is destabilizing in the inviscid case.

If the external gas is supersonic, then the cut-off wave-number is amplitude dependent. However, the cut-off wave-number decreases with increasing amplitude and the growth rate of unstable disturbances decreases with increasing amplitude. In fact, disturbances with wave-numbers near the cut-off wave-number do not grow indefinitely with time, but achieve steady-state amplitude.

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Appendix

$$q = -\frac{1}{8}\frac{M^6}{m^2}(\gamma+1)\left(\frac{\gamma+1}{m^2}-\gamma+3\right). \tag{A 1}$$

$$p_1 = -M^2\left[\left(\frac{\gamma+1}{m^2}-\gamma+3\right)d_{22}+\tfrac{1}{8}(\gamma+1)(2\gamma-3)\frac{M^4}{m^3} + \tfrac{1}{8}(\gamma-1)\frac{M^2}{m}\left(\frac{1}{m^2}+3\right)-\frac{3m^4+2m^2-1}{4m^3}\right]. \tag{A 2}$$

$$p_2 = -\tfrac{1}{2}\mu_1 C_1 a_{22} - \mu_1 b_{22} C_2 - \tfrac{3}{8}\mu_1. \tag{A 3}$$

$$p_3 = \left(2m - \frac{1}{m}\right) d_{22} + \left(\frac{1}{m} - \tfrac{1}{2}m\right) a_{22} - \tfrac{1}{4} - \tfrac{1}{4}(\gamma+1)\frac{M^4}{m^2} - \tfrac{1}{8}m^2. \quad (A\ 4)$$

$$p_4 = (1 - C_1C_2)\mu_1^2 b_{22} - \tfrac{5}{8}C_1\mu_1^2 + \tfrac{3}{8}k^2 + \tfrac{1}{2}a_{22}\mu_1^2. \quad (A\ 5)$$

$$p_5 = m\left[\tfrac{1}{16}(\gamma+1)\frac{M^4}{m^3} - \tfrac{1}{2}a_{22} - \tfrac{5}{8}m - \frac{1}{8}\frac{M^2}{m^2}\left(\frac{3}{m^2}+1\right)\right]. \quad (A\ 6)$$

$$Q = \tfrac{1}{4}q. \quad (A\ 7)$$

$$P_1 = -M^2\left(\frac{d_{22}}{4} + d_{20}\right)\left(\frac{\gamma+1}{m^2} - \gamma + 3\right) - \frac{3}{32}\frac{M^6}{m^3}(\gamma+1)(2\gamma-3)$$
$$-\tfrac{3}{32}(\gamma-1)\frac{M^4}{m}\left(3+\frac{1}{m^2}\right) + M^2\frac{6+9m^4}{16m^3}. \quad (A\ 8)$$

$$P_2 = \tfrac{1}{2}\mu_1 C_1(a_{20} - \tfrac{1}{4}a_{22}) - \tfrac{1}{4}\mu_1 C_2 b_{22} - \tfrac{1}{32}\mu_1. \quad (A\ 9)$$

$$P_3 = \left(\frac{a_{22}}{4} + a_{20}\right)\left(\frac{1}{m} - \tfrac{1}{2}m\right) + \left(\frac{d_{22}}{4} + d_{20}\right)\left(2m - \frac{1}{m}\right) - \tfrac{3}{16} - \tfrac{3}{32}m^2 - \frac{3}{16}\frac{M^4}{m^2}(\gamma+1). \quad (A\ 10)$$

$$P_4 = \left[\tfrac{1}{4}(1 - C_1C_2)b_{22} + \frac{1}{2}\left(\frac{a_{22}}{4} + a_{20}\right) + \tfrac{1}{32}C_1\right]\mu_1^2 + \tfrac{9}{32}k^2. \quad (A\ 11)$$

$$P_5 = m\left[-\frac{1}{2}\left(\frac{a_{22}}{4} + a_{20}\right) + \frac{3}{64}\frac{M^4}{m^3}(\gamma+1) - \tfrac{15}{32}m - \frac{3}{32}\frac{M^2}{m}\left(1 + \frac{3}{m^2}\right)\right]. \quad (A\ 12)$$

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